A Sharp Bound on the Two Variable Power Mean

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Among other things, the familiar power mean satisfies the well known relation

$$\frac{a_1 + \cdots + a_n}{n} \leq \sqrt[k]{\frac{a_1^k + \cdots + a_n^k}{n}}$$

for the arbitrary sequence of positive reals $a_1, \ldots, a_n$ and any positive real $k \geq 1$. This inequality serves as a useful tool that can eliminate unwieldy radicals for our efforts to verify other inequalities. This it does by bounding them *below*. However, a problem arises should we require an upper bound of such a radical. The following result is meant to help address that concern.

Proposition 1 *Let a and b be positive reals and let $k \geq -1$ be an integer. Then*

$$\frac{(1+k)(a-b)^2 + 8ab}{4(a+b)} \geq \left(\frac{a^k + b^k}{2}\right)^{\frac{1}{k}} \qquad (*)$$

with equality where $a = b$ or $k = \pm 1$, and where for $k = 0$ we interpret the right-hand side as $\sqrt{ab}$. Moreover, the inequality also holds for any real number $k \geq 2$. Furthermore, if $1 \leq k \leq 3/2$ or $k \leq -1$, then it holds in the reverse direction.

Proof. For $k = 1$,

$$\frac{(1+1)(a-b)^2 + 8ab}{4(a+b)} = \frac{2a^2 + 4ab + 2b^2}{4(a+b)} = \frac{a+b}{2}$$

For $k = -1$,

$$\frac{(1-1)(a-b)^2 + 8ab}{4(a+b)} = \frac{2}{\frac{1}{b} + \frac{1}{a}} = \left(\frac{a^{-1} + b^{-1}}{2}\right)^{-1}$$

For $k = 0$,

$$\begin{aligned} \frac{a^2 + 6ab + b^2}{4(a+b)} &\geq \sqrt{ab} \\ a^2 + 6ab + b^2 &\geq 4a^{\frac{3}{2}}b^{\frac{1}{2}} + 4a^{\frac{1}{2}}b^{\frac{3}{2}} \\ a^2 - 4a^{\frac{3}{2}}b^{\frac{1}{2}} + 6ab - 4a^{\frac{1}{2}}b^{\frac{3}{2}} + b^2 &\geq 0 \\ (\sqrt{a} - \sqrt{b})^4 &\geq 0 \end{aligned}$$

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Now define $f_p(a,b) = \left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}$. The conclusion of the proof of proposition 1 hinges on the pursuant lemma.

Lemma 1 *The two variable power mean $f_p(a,b)$ is concave in p for $p \geq 1$ and convex in p for $p \leq -1$. That is, $\frac{\partial^2}{\partial p^2}\left[f_p(a,b)\right] \leq 0$ for $p \geq 1$ and $\frac{\partial^2}{\partial p^2}\left[f_p(a,b)\right] \geq 0$ for $p \leq -1$, with equality if and only if $a = b$.*

We first establish the claim for $p = 1$. Observe that $f_p(a,b) = \frac{1}{r} \cdot f_p(ar, br)$. Thus, we may take $b = 1$. We consider the function $F(a)$ given by

$$F(a) := \frac{\partial^2}{\partial p^2}\left[f_p(a,1)\right]\Big|_{p=1} = \ln\left(\frac{a+1}{2}\right) + \frac{1}{2}\left[\ln\left(\frac{a+1}{2}\right)\right]^2 + a\ln\left(\frac{a+1}{2a}\right) + \frac{a}{2}\left[\ln\left(\frac{a+1}{2a}\right)\right]^2$$

In showing that $F(a) \leq 0$ for $a > 0$, the following will be of interest:

$$\begin{aligned}
F_1(a) := \frac{d}{da}\left[F(a)\right] &= \frac{2\ln\left(\frac{a+1}{2}\right) + \ln\left(\frac{a+1}{2a}\right)\left(2a + (a+1)\ln\left(\frac{a+1}{2a}\right)\right)}{2(a+1)} \\
F_2(a) := a(a+1)^2 \cdot \frac{d}{da}\left[F_1(a)\right] &= a\ln\left(\frac{2}{a+1}\right) + \ln\left(\frac{2a}{a+1}\right) \\
F_3(a) := \frac{d}{da}\left[F_2(a)\right] &= -1 + \frac{1}{a} + \ln\left(\frac{2}{1+a}\right)
\end{aligned}$$

Clearly, $F(1) = 0$. Suppose that $F(b_0) = 0$ for some positive real b_0 other than 1. Then, since $F(a)$ is differentiable with respect to a on the positive reals, by Rolle's theorem it must be that for some b_1 strictly between b_0 and 1 we have $F_1(b_1) = 0$. But $F_1(1) = 0$, and since $F_1(a)$ is also differentiable, it must be that for some b_2 strictly between b_1 and 1 we have $\frac{d}{da}\left[F_1(a)\right]\big|_{a=b_2} = 0$, implying that $F_2(b_2) = 0$. Once more by the same idea, there must exist some b_3 strictly between b_2 and 1 for which $F_3(b_3) = 0$. But $F_3(a)$ is a strictly decreasing function of a, so it has at most one positive real root. By inspection, that root is $a = 1$, and so b_3 cannot exist. Therefore, our assumption that b_0 exists was false. It follows that $a = 1$ is the unique positive zero of $F(a)$. Now we compute

$$\begin{aligned}
\lim_{a\to 0^+} F(a) &= \ln(1/2) + 1/2 \cdot \left[\ln(1/2)\right]^2 = \ln(1/2)\left(1 + (1/2)\ln(1/2)\right) < 0 \\
F(2e-1) &= (2e-1)\ln\left(\frac{e}{2e-1}\right)\left(1 + (1/2)\ln\left(\frac{e}{2e-1}\right)\right) < 0
\end{aligned}$$

Therefore, $F(a)$ assumes negative values for $0 < a < 1$ and $a > 1$. Since $F(a)$ is continuous, it follows from the intermediate value theorem that it is nowhere positive. Furthermore, equality holds precisely when $a = 1$, which corresponds directly to $a = b$, as desired.

Now we are ready to generalize p. Observe that $f_{\theta p}(a,b) = f_\theta(a^p, b^p)^{1/p}$. Thus, taking $p \geq 1$ fixed and writing $q = \theta p$, we have

$$p^2 \frac{\partial^2}{\partial q^2}\left[f_q(a,b)\right]\big|_{q=p} = \frac{\partial^2}{\partial \theta^2}\left[f_\theta(a^p,b^p)^{1/p}\right]\Big|_{\theta=1}$$

$$= \frac{1}{p}\left(\frac{1}{p} - 1\right) f_\theta(a^p,b^p)^{\frac{1}{p}-2}\left(\frac{\partial}{\partial\theta}\left[f_\theta(a^p,b^p)\right]\right)^2\Bigg|_{\theta=1} + \frac{1}{p} f_\theta(a^p,b^p)^{\frac{1}{p}-1}\frac{\partial^2}{\partial\theta^2}\left[f_\theta(a^p,b^p)\right]\big|_{\theta=1}.$$

Clearly this is nonpositive. Moreover, if it is zero, then the second term must also be zero, implying that $a = b$. Yet whenever $a = b$, the entire expression is zero, so the lemma is shown for $p \geq 1$.

Now taking $p \leq -1$, we have

$$\begin{aligned} \frac{\partial^2}{\partial p^2}\left[f_p(a,b)\right] &= \frac{\partial^2}{\partial p^2}\left[\frac{1}{f_{-p}(1/a,1/b)}\right] \\ &= \frac{2\left(\frac{\partial}{\partial p}\left[f_{-p}(1/a,1/b)\right]\right)^2 - f_{-p}(1/a,1/b)\frac{\partial^2}{\partial p^2}\left[f_{-p}(1/a,1/b)\right]}{\left(f_{-p}(1/a,1/b)\right)^3} \end{aligned}$$

Since $-p \geq 1$, we have $\frac{\partial^2}{\partial p^2}\left[f_{-p}(1/a,1/b)\right] \leq 0$ with equality if and only if $a = b$. It is easily seen that the second partial is nonnegative for $p \leq -1$ with equality only where $a = b$, as desired. $\square$

It follows from the lemma that $f_2(a,b) - f_1(a,b) \geq \frac{\partial}{\partial p}\left[f_p(a,b)\right]$ for all $p \geq 2$. But

$$\begin{aligned} \frac{(a-b)^2}{4(a+b)} &\geq f_2(a,b) - f_1(a,b) \\ \frac{3a^2+2ab+3b^2}{4(a+b)} &\geq \sqrt{\frac{a^2+b^2}{2}} \\ 3a^2+2ab+3b^2 &\geq (a+b)\sqrt{8a^2+8b^2} \\ (3a^2+2ab+3b^2)^2 &\geq (a+b)^2(8a^2+8b^2) \\ 9a^4+12a^3b+22a^2b^2+12ab^3+9b^4 &\geq 8a^4+16a^3b+16a^2b^2+16ab^3+8b^4 \\ a^4-4a^3b+6a^2b^2-4ab^3+b^4 &= (a-b)^4 \geq 0 \end{aligned}$$

Thus, for $k \geq 2$,

$$\begin{aligned} \frac{(1+k)(a-b)^2+8ab}{4(a+b)} &= f_1(a,b) + (k-1)\left(\frac{(a-b)^2}{4(a+b)}\right) \\ &\geq f_2(a,b) + \int_2^k \frac{\partial}{\partial p}\left[f_p(a,b)\right]dp \\ &= f_k(a,b) \end{aligned}$$

Now we establish the claim for $k = 3/2$. For convenience, put $a = x^2, b = y^2$.

Then

$$\begin{aligned}
\frac{\frac{1}{2}(a-b)^2}{4(a+b)} &\leq f_{3/2}(a,b) - f_1(a,b) \\
5a^2+6ab+5b^2 &\leq 8(a+b)\left(\frac{a^{3/2}+b^{3/2}}{2}\right)^{2/3} \\
5x^4+6x^2y^2+5y^4 &\leq 8(x^2+y^2)\left(\frac{x^3+y^3}{2}\right)^{2/3} \\
4\left(5x^4+6x^2y^2+5y^4\right)^3 &\leq 512(x^2+y^2)^3(x^3+y^3)^2 \\
0 &\leq 512(x^2+y^2)^3(x^3+y^3)^2 - 4\left(5x^4+6x^2y^2+5y^4\right)^3 \\
0 &\leq 4(x-y)^6(3x^6+18x^5y-3x^4y^2+28x^3y^3-3x^2y^4+18xy^5+3y^6)
\end{aligned}$$

Thus, $\frac{\frac{5}{2}(a-b)^2+8ab}{4(a+b)} \leq f_{3/2}(a,b)$ and (*) holds in the reverse direction for $k = 3/2$. But recall that equality holds identically in (*) for $k = 1$. Hence, since $f_k(a,b)$ is concave in k for $k \in [1, 3/2]$ while the left-hand side of (*) is linear in k, it follows that (*) holds in the reverse direction for $1 \leq k \leq 3/2$, as claimed.

Observe that because (*) is homogenous and symmetric with respect to a and b, to prove the claim for $k \leq -1$ it suffices to prove the case $b = 1$. We shall check that $\frac{(a-1)^2}{4(a+1)} \geq \frac{\partial}{\partial p}\left[f_p(a,1)\right]\Big|_{p=-1}$. Calculating the derivative on the right, this is equivalent to

$$\frac{(a-1)^2}{4(a+1)} \geq \frac{2a\left(a\ln(a)+(a+1)\ln\left(\frac{2}{a+1}\right)\right)}{(a+1)^2}$$

We will show that the function $G(a)$ given by

$$G(a) := (a+1)(a-1)^2 - 8a\left(a\ln(a)+(a+1)\ln\left(\frac{2}{a+1}\right)\right)$$

is nonnegative for all $a \geq 1$. In showing this, the following will be of interest:

$$\begin{aligned}
G_1(a) &= G'(a) = (a-1)(3a+1) - 16a\ln(a) - 8(2a+1)\ln\left(\frac{2}{a+1}\right) \\
G_2(a) &= G''(a) = \frac{2}{a+1}\cdot\left((a-1)(3a+5) - 8(a+1)\ln\left(\frac{2a}{a+1}\right)\right) \\
G_3(a) &= G'''(a) = \frac{2(a-1)(3a^2+9a+8)}{a(a+1)^2}
\end{aligned}$$

Clear, $a = 1$ is a zero of $G(a), G_1(a)$ and $G_2(a)$. Now suppose there exists a positive real c_0 other than 1 such that $G(c_0) = 1$. Then by Rolle's theorem, there exists a number c_1 strictly between c_0 and 1 such that $G_1(c_1) = 0$. Since $G_1(1) = 0$, by the same principle there must exist c_2 strictly between c_1 and 1 such that $G_2(c_2) = 0$. Likewise, there exists c_3 strictly between c_2 and 1 such that $G_3(c_3) = 0$. Since by inspection $a = 1$ is the unique real root of $G_3(a) = 0$, g_3 cannot exist, which is a contradiction. Hence, $a = 1$ is the unique root of $G(a) = 0$. It is easily seen that $\lim_{a\to 0^+} G(a) = 1$ and that $G(a)$ grows unbounded as a tends to infinity. Hence, by the intermediate value theorem, $G(a)$ can never be negative.

Combining this fact with the lemma, we deduce that $\frac{(a-b)^2}{4(a+b)} \geq \frac{\partial}{\partial p}\left[f_p(a,b)\right]$ for all $p \leq -1$. Therefore,

$$\begin{aligned} 0 &\geq \int_p^{-1} \left(\frac{\partial}{\partial p}\left[f_p(a,b)\right] - \frac{(a-b)^2}{4(a+b)} \right) dp \\ &= f_{-1}(a,b) - f_p(a,b) + \frac{(p+1)(a-b)^2}{4(a+b)} \\ f_p(a,b) &\geq \frac{(1+p)(a-b)^2 + 8ab}{4(a+b)} \end{aligned}$$

completing the proof of proposition 1. †

Although we do not show it here, the difference between the two sides behaves asymptotically as $C_1(a-b)^4$ near equality, excepting in the case $k = 3/2$, where the difference converges to 0 as $C_2(a-b)^6$.

Can the proposition be generalized to n variables? While it may be possible to conjecture a generalized radical-free expression, in the author's opinion a proof will likely be considerably more difficult. In particular, there does not exist a lower bound P such that the arbitrary power mean $g_p(a_1, \dots, a_n) = \left(\frac{a_1^p + \cdots + a_n^p}{n}\right)^{1/p}$ satisfies $\frac{\partial^2}{\partial p^2}\left[g_p(a_1, \dots, a_n)\right] \leq 0$ for all $p \geq P$. Consider the sequence $x_1, \dots, x_n$ given by $x_i = \frac{i}{n}$ for $i = 1, \dots, n$. Employing a Riemann sum, we write

$$\psi_\lambda(p) = \lim_{n\to\infty} g_p(x_1^\lambda, \dots, x_n^\lambda) = \lim_{n\to\infty} \sqrt[p]{\frac{\sum_{i=1}^n x_i^{\lambda p}}{n}} = \left(\int_0^1 x^{\lambda p} dx\right)^{\frac{1}{p}} = (1+\lambda p)^{-\frac{1}{p}}$$

We compute

$$\begin{aligned} \frac{\partial}{\partial p}\left[\psi_\lambda(p)\right] &= \left(\frac{1}{p^2}\ln(1+\lambda p) - \frac{\lambda}{p(1+\lambda p)}\right)\psi_\lambda(p) \\ \frac{\partial^2}{\partial p^2}\left[\psi_\lambda(p)\right] &= \left(\left(\frac{1}{p^2}\ln(1+\lambda p) - \frac{\lambda}{p(1+\lambda p)}\right)^2 - \frac{2}{p^3}\ln(1+\lambda p) + \frac{2\lambda + 3\lambda^2 p}{p^2(1+\lambda p)^2}\right)\psi_\lambda(p) \end{aligned}$$

It follows that for a given interval (a,b), $\psi_\lambda(p)$ becomes convex with respect to $p \in (a,b)$ for sufficiently large λ, since one easily checks that the dominant term in the second partial is $\frac{1}{p^4}\left(\log(1+\lambda p)\right)^2$. A generalization to n variables would therefore require at least one novel idea to circumvent the collapse of lemma 1. It remains to be seen whether this is feasible.